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# GENERALIZED STRICHARTZ INEQUALITIES FOR THE WAVE EQUATION ON THE LAGUERRE HYPERGROUP

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**Abstract**

In this paper we study generalized Strichartz inequalities for the wave equation on the Laguerre hypergroup using generalized homogeneous Besov-Laguerre type spaces.

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## 0. Introduction

Let $\alpha \geq 0$ and $\mathbb{K} = [0, +\infty[ \times \mathbb{R}$. We consider the generalized wave equation operators

$$\Box_{\mathbb{K}} = \partial_t^2 - D_{\mathbb{K}}$$

in the two space variables $(y, s) \in \mathbb{K}$ and time $t \in \mathbb{R}$ where

$$D_{\mathbb{K}} := \frac{\partial^2}{\partial y^2} + \frac{2\alpha + 1}{y}\frac{\partial}{\partial y} + y^2 \frac{\partial^2}{\partial s^2}\ ; \qquad (y, s) \in ]0, \infty[ \times \mathbb{R}.$$

Let $f \in L^1([0, T], L^2(\mathbb{K}))$ and $(u_0, u_1) \in \dot{\mathcal{H}}^1(\mathbb{K}) \times L^2(\mathbb{K})$; $\dot{\mathcal{H}}^1(\mathbb{K})$ is being the homogeneous Sobolev space on the Laguerre hypergroup. It is well known that the Cauchy problem

$$\left\{ \begin{array}{rcl} \Box_{\mathbb{K}} u &=& f, \\ (u|_{t=0}, \partial_t u|_{t=0}) &=& (u_0, u_1) \end{array} \right. \tag{1}$$

is well posed in the energy space $(u, \partial_t u) \in \mathcal{C}([0,T], \dot{\mathcal{H}}^1(\mathbb{K}) \times L^2(\mathbb{K}))$ and that the energy

$$E(u) = \left\|u(.,t)\right\|^2_{\dot{\mathcal{H}}^1(\mathbb{K})} + \left\|\partial_t u(.,t)\right\|^2_{L^2(\mathbb{K})}$$

is constant independent of $t$ for solutions of (1). For $\alpha = n-1$; $n \in \mathbb{N}\backslash\{0\}$, the operator $D_{\mathbb{K}}$ is the radial part of the sub Laplacian on the Heisenberg group $\mathbb{H}^n$. We denote by $\varphi_{\lambda,m}$; $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$, the unique solution of the following system:

$$\begin{cases} \partial_s u = i\lambda u, \\ D_{\mathbb{K}} u = -4|\lambda|(m + \frac{\alpha+1}{2})u; \\ u(0,0) = 1, \ \dfrac{\partial u}{\partial y}(0,s) = 0 \qquad \text{for all } \ s \in \mathbb{R}. \end{cases}$$

One knows that $\varphi_{\lambda,m}(y,s) = e^{i\lambda s}\mathcal{L}_m^\alpha(|\lambda| y^2)$, where $\mathcal{L}_m^\alpha$ is the Laguerre functions defined on $\mathbb{R}_+$ by $\mathcal{L}_m^\alpha(x) = e^{-\frac{x}{2}} \frac{L_m^\alpha(x)}{L_m^\alpha(0)}$ and $L_m^\alpha$ is the Laguerre polynomial of degree $m$ and order $\alpha$ ([16], [8], [10], [15]). We recall that for $(\lambda, m) \in \mathbb{R}\times\mathbb{N}$ and for a suitable function $f : \mathbb{K} \longrightarrow \mathbb{C}$ the Fourier-Laguerre transform $\mathcal{F}(f)(\lambda, m)$ of $f$ at $(\lambda, m)$ is defined by ([17], [2], [20, 21], [9]):

$$\mathcal{F}(f)(\lambda, m) = \int_{\mathbb{K}} \varphi_{-\lambda,m}(y,s) f(y,s) d\mu_\alpha(y,s), \tag{2}$$

where $d\mu_\alpha(y,s) = \dfrac{y^{2\alpha+1} dy ds}{\pi\Gamma(\alpha+1)}$.

It has been proved in [17, Theorem II.1] that the Fourier-Laguerre transform is a topological isomorphism from $S_*(\mathbb{K})$ onto $S(\mathbb{R}\times\mathbb{N})$, where

- $S_*(\mathbb{K})$ is the Schwartz space of functions $\psi : \mathbb{R}^2 \longrightarrow \mathbb{C}$ even with respect to the first variable, $\mathcal{C}^\infty$ on $\mathbb{R}^2$ and rapidly decreasing together with all their derivatives; i.e. for all $k, p, q \in \mathbb{N}$ we have

$$\widetilde{\mathcal{N}}_{k,p,q}(\psi) = \sup_{(y,s)\in\mathbb{K}} \left\{(1 + y^2 + s^2)^k \left|\frac{\partial^{p+q}}{\partial y^p \partial s^q} \psi(y,s)\right|\right\} < \infty. \tag{3}$$

- $\mathcal{S}(\mathbb{R}\times\mathbb{N})$ the space of functions $\Psi : \mathbb{R}\times\mathbb{N} \longrightarrow \mathbb{C}$ satisfying:

  **i)** For all $m, p, q, r, s \in \mathbb{N}$, the function

$$\lambda \longmapsto \lambda^p \Big(|\lambda|(m + \frac{\alpha+1}{2})\Big)^q \Lambda_1^r \Big(\Lambda_2 + \frac{\partial}{\partial\lambda}\Big)^\gamma \Psi(\lambda, m)$$

is bounded and continuous on $\mathbb{R}$, $\mathcal{C}^\infty$ on $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ and such that the left and the right derivatives at zero exist.

**ii )** For all $k, p, q \in \mathbb{N}$, we have

$$\mathcal{V}_{k,p,q}(\Psi) = \sup_{(\lambda,m)\in\mathbb{R}^*\times\mathbb{N}} \left\{ \left(1+\lambda^2(1+m^2)\right)^k \left| \Lambda_1^p \Big(\Lambda_2 + \frac{\partial}{\partial\lambda}\Big)^q \Psi(\lambda, m)\right| \right\} < \infty, \tag{4}$$

where

- $\Lambda_1\Psi(\lambda, m) = \frac{1}{|\lambda|}\Big(m\Delta_+\Delta_-\Psi(\lambda, m) + (\alpha+1)\Delta_+\Psi(\lambda, m)\Big)$.
- $\Lambda_2\Psi(\lambda, m) = \frac{-1}{2\lambda}\Big((\alpha+m+1)\Delta_+\Psi(\lambda, m) + m\Delta_-\Psi(\lambda, m)\Big)$.
- $\Delta_+\Psi(\lambda, m) = \Psi(\lambda, m+1) - \Psi(\lambda, m)$.
- $\Delta_-\Psi(\lambda, m) = \Psi(\lambda, m) - \Psi(\lambda, m-1), \quad \text{if} \quad m \geq 1$ and $\Delta_-\Psi(\lambda, 0) = \Psi(\lambda, 0)$.

We note that $S_*(\mathbb{K})$ (resp. $S(\mathbb{R}\times\mathbb{N})$) equipped with the semi-norms $\widetilde{\mathcal{N}}_{k,p,q}$ (resp. $\mathcal{V}_{k,p,q}$), $k, p, q \in \mathbb{N}$, is a Fréchet space ([17]).

This paper is organized as follows: In the first section we collect some harmonic analysis results on the Laguerre hypergroup which are developed in [17] and [20, 21]. In the second section we recall the definition and some properties of generalized homogeneous Besov-Laguerre type spaces $\dot{\Lambda}^\gamma_{p,q}(\mathbb{K})$; $1 \leq p, q \leq \infty$, $\gamma \in \mathbb{R}$, similar to the classical one's given in [7], [5], [6] and [23]. These spaces are introduced in terms of convolution of tempered distributions with a class of smooth functions. It has been proved in [1] that these spaces are Banach spaces for $\gamma < (2\alpha+4)/p$. We prove that $\dot{\Lambda}^\gamma_{2,2}(\mathbb{K})$ coincide with the homogeneous Sobolev-Laguerre type space $\dot{\mathcal{H}}^\gamma(\mathbb{K})$. In the third section we generalize, for solutions of Cauchy problem (1), the Strichartz inequalities (see [22], [14] and [3]) to the Laguerre hypergroup. Many functional analysis results which remained valid also in our context are used to prove the main results given in Theorems 3.1 and 3.2.

Finally, we mention that, $C$ will be always used to denote a suitable positive constant that is not necessarily the same in each occurrence.

## 1. Preliminaries

Throughout this paper we fix $\alpha \geq 0$ and we denote by

- $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ and $\mathbb{R}^*_+ = ]0, +\infty[$.

- $\mathcal{C}_*(\mathbb{K})$ the space of continuous functions on $\mathbb{R}^2$ even with respect to the first variable.

- $\mathcal{C}_{*,c}(\mathbb{K})$ the subspace of $\mathcal{C}_*(\mathbb{K})$ consisting of functions with compact support.

- $S^1_{*,0}(\mathbb{K})$ the subset of functions $\psi$ in $S_*(\mathbb{K})$ such that $\mathcal{F}\psi \in \mathcal{D}(\mathbb{R}^* \times \mathbb{N})$ and
$$\int_0^\infty \Big(\mathcal{F}\psi(r^2\lambda, m)\Big)^2 \frac{dr}{r} = 1, \qquad \text{for } (\lambda, m) \in \mathbb{R}^* \times \mathbb{N}.$$
These functions are known as generalized wavelets on $\mathbb{K}$ ([17]).

- $\mathcal{D}(\mathbb{R} \times \mathbb{N})$ the subspace of $S(\mathbb{R} \times \mathbb{N})$ of functions $\psi$ satisfying the following:

  i) There exists $m_0 \in \mathbb{N}$ satisfying $\psi(\lambda, m) = 0$, for all $(\lambda, m) \in \mathbb{R}\times\mathbb{N}$ such that $m > m_0$.
  ii) For all $m \leq m_0$, the function $\lambda \longmapsto \psi(\lambda, m)$ is $\mathcal{C}^\infty$ on $\mathbb{R}$, with compact support and vanishes in a neighborhood of zero.

- $L^p(\mathbb{K}) = L^p(\mathbb{K}, d\mu_\alpha)$, $1 \leq p \leq \infty$, the space of Borel measurable functions on $\mathbb{K}$ such that $\|f\|_{L^p(\mathbb{K})} < \infty$, where
$$\begin{aligned} \|f\|_{L^p(\mathbb{K})} &= \left(\int_{\mathbb{K}} |f(y,s)|^p d\mu_\alpha(y,s)\right)^{\frac{1}{p}}, \qquad \text{if} \quad p \in [1, \infty[, \\ \|f\|_{L^\infty(\mathbb{K})} &= \operatorname*{ess\,sup}_{(y,s)\in\mathbb{K}} |f(y,s)|, \end{aligned}$$
$d\mu_\alpha$ being the positive measure defined on $\mathbb{K}$ given in the introduction.

Each of these spaces is equipped with its usual topology.

Definition 1.1.

- The generalized translation operators $T^{(\alpha)}_{(y,s)}$ on the Laguerre hypergroup are given for a suitable function $f$ by:

$$T^{(\alpha)}_{(y,s)} f(x,z) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f((x^2+y^2+2xy\cos\theta)^{\frac{1}{2}}, z+s+xy\sin\theta)d\theta, \\ \qquad \text{if } \alpha = 0, \\ \dfrac{\alpha}{\pi}\displaystyle\int_0^1\int_0^{2\pi} f((x^2+y^2+2xy\rho\cos\theta)^{\frac{1}{2}}, z+s+xy\rho\sin\theta) \\ \qquad \times \rho(1-\rho^2)^{\alpha-1} d\theta d\rho, \qquad \text{if } \alpha > 0. \end{cases}$$

• The generalized convolution product on the Laguerre hypergroup is defined for a suitable pair of functions $f$ and $g$ by:

$$f_{\#}g(y,s) = \int_{\mathbb{K}} T^{(\alpha)}_{(y,s)} f(x,z) g(x,-z) d\mu_\alpha(x,z) \qquad \text{for all} \quad (y,s) \in \mathbb{K}.$$

We recall that $(\mathbb{K}, {}_{\#}, i)$ is an hypergroup in the sense of Jewett ([13], [4]), where $i$ denotes the involution defined on $\mathbb{K}$ by $i(y,s) = (y,-s)$. This hypergroup is the Laguerre hypergroup which coincides, for $\alpha = n-1$; $n \in \mathbb{N}\backslash\{0\}$, with the hypergroup of radial functions on the Heisenberg group (see [17]).

**Notations.** Let $r > 0$. We will denote by

- $(y,s)_r = (\frac{y}{r}, \frac{s}{r^2})$ the dilation of $(y,s) \in \mathbb{K}$.
- $f_r(y,s) = r^{-(2\alpha+4)} f((y,s)_r)$ the dilation of the function $f$ defined on $\mathbb{K}$ preserving the mean of $f$ with respect to the measure $d\mu_\alpha$, in the sense that

$$\int_{\mathbb{K}} f_r(y,s) d\mu_\alpha(y,s) = \int_{\mathbb{K}} f(y,s) d\mu_\alpha(y,s), \quad \text{for all} \ \ r > 0 \text{ and } f \in L^1(\mathbb{K}). \tag{5}$$

Proposition 1.1. *The following properties hold ([17])*

1) *Let $f$ be in $L^p(\mathbb{K})$, $1 \leq p \leq \infty$. Then for all $(y,s) \in \mathbb{K}$, the function $T^{(\alpha)}_{(y,s)} f$ belongs to $L^p(\mathbb{K})$ and we have*

$$\|T^{(\alpha)}_{(y,s)} f\|_{L^p(\mathbb{K})} \leq \|f\|_{L^p(\mathbb{K})}.$$

2) *For $f$ in $L^p(\mathbb{K})$ and $g$ in $L^q(\mathbb{K})$, $1 \leq p,q \leq \infty$, the function $f_{\#}g$ belongs to $L^r(\mathbb{K})$; $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, and we have*

$$\|f_{\#}g\|_{L^r(\mathbb{K})} \leq \|f\|_{L^p(\mathbb{K})} \|g\|_{L^q(\mathbb{K})}.$$

3) (i) *Let $f$ be in $L^1(\mathbb{K})$. Then the function $\mathcal{F}(f)$ is bounded on $\mathbb{R} \times \mathbb{N}$ and we have*

$$\|\mathcal{F}(f)\|_{L^\infty(\mathbb{R}\times\mathbb{N})} \leq \|f\|_{L^1(\mathbb{K})}$$

*where* $\|\mathcal{F}(f)\|_{L^\infty(\mathbb{R}\times\mathbb{N})} = \operatorname*{ess\,sup}_{(\lambda,m)\in\mathbb{R}\times\mathbb{N}} |\mathcal{F}(f)(\lambda,m)|.$

*(ii) Let $f$ and $g$ in $L^1(\mathbb{K})$, then we have*

$$\mathcal{F}(f_{\#}g) = \mathcal{F}(f)\mathcal{F}(g).$$

*(iii) Let $f$ be in $L^1(\mathbb{K})$. Then for all $(y,s)$ in $\mathbb{K}$ and $(\lambda,m)$ in $\mathbb{R}\times\mathbb{N}$, we have*

$$\mathcal{F}(T^{(\alpha)}_{(y,s)}f)(\lambda,m) = \varphi_{\lambda,m}(y,s)\mathcal{F}(f)(\lambda,m).$$

## 2. Generalized homogeneous Besov-Laguerre type spaces

In this section we summarize some results on the generalized homogeneous Besov-Laguerre type spaces studied in [1].

Definition 2.1. Let $1 \le p,q \le \infty$, $\gamma \in \mathbb{R}$ and $\psi \in S^1_{*,0}(\mathbb{K})$. We define the generalized homogeneous Besov-Laguerre type spaces $\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})$ as the set of tempered distributions $f$ such that

$$f = \int_0^\infty f_{\#}\psi_{r\#}\psi_r \frac{dr}{r} \tag{6}$$

and $\|f\|_{\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})} < \infty$, where

$$\|f\|_{\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})} = \begin{cases} \left(\displaystyle\int_0^\infty \Big(\frac{\|f_{\#}\psi_r\|_p}{r^\gamma}\Big)^q \frac{dr}{r}\right)^{\frac{1}{q}}, & \text{if} \quad 1 \le q < \infty, \\ \operatorname*{ess\,sup}_{r>0} \Big(\dfrac{\|f_{\#}\psi_r\|_p}{r^\gamma}\Big), & \text{if} \quad q = \infty. \end{cases}$$

Remark 2.1.

**1)** We begin by mentioning that the definition of the generalized homogeneous Besov-Laguerre type spaces given here is the same than that introduced by Chemin in the classical case (see [7]) and generalized by Bahouri, Gérard and Xu on the Heisemberg group (see [3]). We do not choose the classical definition introduced by Peetre (see [19]) in which $\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})$ is defined as a set of distributions modulo polynomials. In fact in the case $\gamma < \frac{2\alpha+4}{p}$, the condition $\|f\|_{\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{K})} < \infty$ implies the convergence of the integral

$$\int_0^\infty f_{\#}\psi_{r\#}\psi_r \frac{dr}{r}$$

in the sense of distributions and not only in the sense of distributions modulo polynomials, thus the two points of view are equivalent and that the expression (6) is independent, in $S'_*(\mathbb{K})$, of the choice of $\psi$ in $S^1_{*,0}$. We note finally that, similarly to the classical case, for $\gamma \geq \frac{2\alpha+4}{p}$, the space $\overset{\bullet}{\Lambda}{}^{\gamma}_{p,q}(\mathbb{K})$, as we define, is not Banach space.

**2)** If $f$ belongs to $L^2(\mathbb{K})$, then (6) holds in $L^2(\mathbb{K})$. Which is a consequence of Plancherel's formula (see [18]). Hence one can write

$$\left\| f - \int_{1/\varepsilon}^{\varepsilon} f \# \psi_r \# \psi_r \frac{dr}{r} \right\|_2^2$$

$$= \int_{-\infty}^{+\infty} \left\{ \sum_{m=0}^{+\infty} L_m^\alpha(0) \Big| \mathcal{F} f(\lambda, m) \Big|^2 \Big| 1 - \int_{1/\varepsilon}^{\varepsilon} \Big( \mathcal{F}\psi_r(\lambda, m) \Big)^2 \frac{dr}{r} \Big|^2 \right\} |\lambda|^{\alpha+1} d\lambda.$$

And, using Lebesgue theorem, the right hand side of the above equality tends to zero as $\varepsilon$ tends to $+\infty$. Indeed,

$$\Big| 1 - \int_{1/\varepsilon}^{\varepsilon} \Big( \mathcal{F}\psi_r(\lambda, m) \Big)^2 \frac{dr}{r} \Big|^2 \longrightarrow 0 \qquad \text{as } \varepsilon \longrightarrow +\infty$$

and

$$\sum_{m=0}^{+\infty} L_m^\alpha(0) \Big| \mathcal{F} f(\lambda, m) \Big|^2 \Big| 1 - \int_{1/\varepsilon}^{\varepsilon} \Big( \mathcal{F}\psi_r(\lambda, m) \Big)^2 \frac{dr}{r} \Big|^2$$
$$\leq \sum_{m=0}^{+\infty} L_m^\alpha(0) \Big| \mathcal{F} f(\lambda, m) \Big|^2 \in L^1(\mathbb{R}, |\lambda|^{\alpha+1} d\lambda).$$

**3)** The expression (6) is not true in $S'_*(\mathbb{K})$ if $f$ is a polynomial function on $\mathbb{K}$. Indeed in this case, for all $r > 0$, we have $f \# \psi_r = 0$.

**4)** For $1 \leq p, q \leq \infty$ and $\gamma \in \mathbb{R}$, the space $\overset{\bullet}{\Lambda}{}^{\gamma}_{p,q}(\mathbb{K})$ is independent of the choice of the function $\psi$ in $S^1_{*,0}(\mathbb{K})$.

**5)** For $1 \leq p, q \leq \infty$ and $\gamma \in \mathbb{R}$, the Besov-Laguerre type space $\overset{\bullet}{\Lambda}{}^{\gamma}_{p,q}(\mathbb{K})$ is homogeneous of degree $d(p, \gamma) = \frac{2\alpha+4}{p} - \gamma$ in the sense that, for all $f \in \overset{\bullet}{\Lambda}{}^{\gamma}_{p,q}(\mathbb{K})$

$$\|d_r f\|_{\overset{\bullet}{\Lambda}{}^{\gamma}_{p,q}(\mathbb{K})} = r^{\frac{2\alpha+4}{p} - \gamma} \|f\|_{\overset{\bullet}{\Lambda}{}^{\gamma}_{p,q}(\mathbb{K})}, \quad \text{for all} \quad r > 0,$$

where $d_r f(y, s) = f\big((y, s)_r\big)$, for all $(y, s) \in \mathbb{K}$.

In what follows we collect some properties of the generalized homogeneous Laguerre-Besov type-spaces. For more details, see [1].

Proposition 2.1. *Let $1 \leq p, q \leq \infty$ and $\gamma < \frac{2\alpha+4}{p}$. Then $\overset{\bullet}{\Lambda}{}^{\gamma}_{p,q}(\mathbb{K})$ is a Banach space, and the result remains also valid for $\gamma = \frac{2\alpha+4}{p}$ if $q = 1$.*

Proposition 2.2.

**1)** *Let $1 \leq q \leq \infty$ and $\gamma_1, \gamma_2 \in \mathbb{R}$. Then, for $1 \leq p_1 \leq p_2 \leq \infty$ such that $d(p_1, \gamma_1) = d(p_2, \gamma_2)$, we have*

$$\overset{\bullet}{\Lambda}{}^{\gamma_1}_{p_1,q}(\mathbb{K}) \subseteq \overset{\bullet}{\Lambda}{}^{\gamma_2}_{p_2,q}(\mathbb{K}) \qquad \textit{(with continuous embedding).}$$

**2)** *Let $2 \leq p \leq \infty$. Then*

$$\overset{\bullet}{\Lambda}{}^{0}_{p,2}(\mathbb{K}) \subseteq L^p(\mathbb{K}) \qquad \textit{(with continuous embedding).}$$

**3)** *Let $1 \leq p \leq \infty$. Then we have*

$$\overset{\bullet}{\Lambda}{}^{0}_{p,1}(\mathbb{K}) \subseteq L^p(\mathbb{K}) \qquad \textit{(with continuous embedding).}$$

**4)** *For all $1 \leq p \leq \infty$, $\gamma \in \mathbb{R}$ and $1 \leq q_1 \leq q_2 \leq \infty$*

$$\overset{\bullet}{\Lambda}{}^{\gamma}_{p,q_1}(\mathbb{K}) \subseteq \overset{\bullet}{\Lambda}{}^{\gamma}_{p,q_2}(\mathbb{K}) \qquad \textit{(with continuous embedding).}$$

Remark 2.2. It is clear that $\overset{\bullet}{\Lambda}{}^{0}_{2,2}(\mathbb{K}) = L^2(\mathbb{K})$. More general, we prove that $\overset{\bullet}{\Lambda}{}^{\gamma}_{2,2}(\mathbb{K})$ is a Hilbert space which coincides with the homogeneous Sobolev-Laguerre type space $\overset{\bullet}{\mathcal{H}}{}^{\gamma}(\mathbb{K})$ defined as follows:

Definition 2.2. Let $\gamma \in \mathbb{R}$. The homogeneous Sobolev-Laguerre type space $\overset{\bullet}{\mathcal{H}}{}^{\gamma}(\mathbb{K})$ is the set of all tempered distributions $f$ such that $\mathcal{F}f \in L^2_{loc}(\mathbb{K})$ and

$$\|f\|_{\overset{\bullet}{\mathcal{H}}{}^{\gamma}(\mathbb{K})} = \left( \sum_{m=0}^{+\infty} L_m^\alpha(0) \int_{\mathbb{R}} \left( |\lambda| \left( m + \frac{\alpha+1}{2} \right) \right)^{\gamma} \left| \mathcal{F}f(\lambda, m) \right|^2 |\lambda|^{\alpha+1} d\lambda \right)^{1/2} < +\infty.$$

Theorem 2.1. *Let $\gamma < \alpha + 2$. The homogeneous Besov-Laguerre type space $\overset{\bullet}{\Lambda}{}^{\gamma}_{2,2}(\mathbb{K})$ is equal to the homogeneous Sobolev-Laguerre type space $\overset{\bullet}{\mathcal{H}}{}^{\gamma}(\mathbb{K})$ with equivalent norms.*

P r o o f. It suffices to prove that there exist $C_1, C_2 > 0$, such that, for all $f \in S'_*(\mathbb{K})$,

$$C_1 \|f\|_{\overset{\bullet}{\Lambda}{}^{\gamma}_{2,2}(\mathbb{K})} \leq \|f\|_{\overset{\bullet}{\mathcal{H}}{}^{\gamma}(\mathbb{K})} \leq C_2 \|f\|_{\overset{\bullet}{\Lambda}{}^{\gamma}_{2,2}(\mathbb{K})}.$$

By Plancherel's theorem (see [18]) one has

$$\|f\|^2_{\dot{\Lambda}^{\gamma}_{2,2}(\mathbb{K})} = \sum_{m=0}^{+\infty} L_m^{\alpha}(0) \int_{\mathbb{R}} \Big|\mathcal{F}f(\lambda,m)\Big|^2 \int_0^{+\infty} \frac{|\mathcal{F}\psi_r(\lambda,m)|^2}{r^{2\gamma}} \frac{dr}{r} |\lambda|^{\alpha+1} d\lambda$$

and taking into account that

$$\|f\|^2_{\dot{\mathcal{H}}^{\gamma}(\mathbb{K})} = \sum_{m=0}^{+\infty} L_m^{\alpha}(0) \int_{\mathbb{R}} \Big(|\lambda|\Big(m+\frac{\alpha+1}{2}\Big)\Big)^{\gamma} \Big|\mathcal{F}f(\lambda,m)\Big|^2 |\lambda|^{\alpha+1} d\lambda$$

we have to compare

$$\int_0^{+\infty} \frac{|\mathcal{F}\psi_r(\lambda,m)|^2}{r^{2\gamma}} \frac{dr}{r} \qquad \text{and} \qquad \Big(|\lambda|\Big(m+\frac{\alpha+1}{2}\Big)\Big)^{\gamma}.$$

From the hypothesis $\psi \in S^1_{*,0}$, there exist $a, b, A, B > 0$ such that

$$Supp\mathcal{F}\psi \subset \Big\{(\lambda,m) \in \mathbb{R}\times\mathbb{N}; \quad a \leq |\lambda|\Big(m+\frac{\alpha+1}{2}\Big) \leq b\Big\}$$

and

$$A \leq |\mathcal{F}\psi_r(\lambda,m)|^2 \leq B \qquad \text{on } Supp\mathcal{F}\psi_r.$$

This leads to

$$\int_0^{+\infty} \frac{|\mathcal{F}\psi_r(\lambda,m)|^2}{r^{2\gamma}} \frac{dr}{r} = \int_{a(|\lambda|(m+\frac{\alpha+1}{2}))^{-1/2}}^{b(|\lambda|(m+\frac{\alpha+1}{2}))^{-1/2}} \frac{|\mathcal{F}\psi(r^2\lambda,m)|^2}{r^{2\gamma}} \frac{dr}{r}.$$

So the desired inequalities hold. ∎

## 3. Generalized Strichartz inequalities on the Laguerre hypergroup

It is well known that the Cauchy problem (1) is solved by $u = v + w$ where $v$ is the solution of the homogeneous equation with the same data

$$\begin{cases} \partial_t^2 u - D_{\mathbb{K}} u &= 0 \\ (u|_{t=0}, \partial_t u|_{t=0}) &= (u_0, u_1) \end{cases} \tag{7}$$

and $w$ is the solution of the inhomogeneous equation with zero, data

$$\begin{cases} \partial_t^2 u - D_{\mathbb{K}} u &= f \\ (u|_{t=0}, \partial_t u|_{t=0}) &= (0, 0). \end{cases} \tag{8}$$

Following closely the arguments of Ginibre-Velo [11], one can prove, as in [3], the following lemma that will be useful in the sequel.

Lemma 3.1. *Let $\gamma_1, \gamma_2 \in \mathbb{R}$ and $p_1, p_2, r_1, r_2 \in [2, +\infty]$ satisfying:*

$$2/p_i + 1/r_i \leq 1/2, \qquad \text{for} \quad i = 1, 2, \tag{9}$$

$$1/p_1 + (2\alpha+4)/r_1 - \gamma_1 = (2\alpha+4)/2 - 1, \tag{10}$$

$$1/p_2 + (2\alpha+4)/r_2 - \gamma_2 = (2\alpha+4)/2. \tag{11}$$

**1)** *For all $(u_0, u_1) \in \dot{\mathcal{H}}^1(\mathbb{K}) \times L^2(\mathbb{K})$, we have*

$$\|v\|_{L^{p_1}(\mathbb{R},\, \dot{\Lambda}^{\gamma_1}_{r_1,2}(\mathbb{K}))} + \|\partial_t v\|_{L^{p_1}(\mathbb{R},\, \dot{\Lambda}^{\gamma_1-1}_{r_1,2}(\mathbb{K}))} \leq C\Big\{\|u_0\|_{\dot{\mathcal{H}}^1(\mathbb{K})} + \|u_1\|_{L^2(\mathbb{K})}\Big\}. \tag{12}$$

**2)** *For all interval $I$ containing $0$, we have*

$$\|w\|_{L^{p_1}(I,\, \dot{\Lambda}^{\gamma_1}_{r_1,2}(\mathbb{K}))} + \|\partial_t w\|_{L^{p_1}(I,\, \dot{\Lambda}^{\gamma_1-1}_{r_1,2}(\mathbb{K}))} \leq C\|f\|_{L^{\bar{p}_2}(I,\, \dot{\Lambda}^{-\gamma_2}_{\bar{r}_2,2}(\mathbb{K}))}, \tag{13}$$

*where $\bar{p}_2$ and $\bar{r}_2$ are the conjugate exponents of $p_2$ and $r_2$ respectively.*

The main results in this work are the generalized Strichartz inequalities given in the following two theorems.

Theorem 3.1. *Let $p \in [4\alpha+3, +\infty]$ and let $q$ such that $1/p + (2\alpha+4)/q = \alpha+1$. Then there exists $C_q > 0$ such that, for all $T > 0$, we have*

$$\|u\|_{L^p([0,T],L^q(\mathbb{K}))} \leq C_q \left[\|f\|_{L^1([0,T],L^2(\mathbb{K}))} + E_0^{1/2}(u)\right],$$

*where $E_0(u) = \|u_0\|^2_{\dot{\mathcal{H}}^1(\mathbb{K})} + \|u_1\|^2_{L^2(\mathbb{K})}$.*

P r o o f. First, from the conditions on $p$ and $q$, one can verify that $q \geq 2$. And hence, from Proposition 2.2, we have $\dot{\Lambda}^0_{q,2}(\mathbb{K}) \subset L^q(\mathbb{K})$. That is

$$\|v\|_{L^p(I,L^q(\mathbb{K}))} \leq C\|v\|_{L^p(I,\, \dot{\Lambda}^0_{q,2}(\mathbb{K}))}.$$

Using Lemma 3.1, we obtain

$$\begin{aligned} \|v\|_{L^p(I,L^q(\mathbb{K}))} &\leq C\Big\{\|v\|_{L^{p_1}(I,\, \dot{\Lambda}^0_{q,2}(\mathbb{K}))} + \|\partial_t v\|_{L^{p_1}(I,\, \dot{\Lambda}^{-1}_{q,2}(\mathbb{K}))}\Big\} \\ &\leq C\{\|u_0\|_{\dot{\mathcal{H}}^1(\mathbb{K})} + \|u_1\|_{L^2(\mathbb{K})}\} \\ &\leq C E_0^{1/2}(u). \end{aligned}$$

To obtain the estimate on $w$, we take $p_1 = p$, $r_1 = q$, $\gamma_1 = 0$, $p_2 = \infty$, $r_2 = 2$ and $\gamma_2 = 0$ in (13), then it holds

$$\begin{aligned}
\|w\|_{L^p(I,L^q(\mathbb{K}))} &\leq C\|w\|_{L^p(I,\, \overset{\bullet}{\Lambda}{}^0_{q,2}(\mathbb{K}))} \\
&\leq C\Big\{\|w\|_{L^{p_1}(I,\, \overset{\bullet}{\Lambda}{}^0_{q,2}(\mathbb{K}))} + \|\partial_t w\|_{L^{p_1}(I,\, \overset{\bullet}{\Lambda}{}^{-1}_{q,2}(\mathbb{K}))}\Big\} \\
&\leq C\|f\|_{L^{\bar{p}_2}(I,\, \overset{\bullet}{\Lambda}{}^{-\gamma_2}_{\bar{r}_2,2}(\mathbb{K}))}.
\end{aligned}$$

The required estimate is proved. ■

Our purpose now is to prove that the method given by Ginibre and Velo in [11] and by Bahouri, Gérard and Xu in [3] can be generalized to the Laguerre hypergroup. Using a suitable representation of the solution $u$ we prove decreasing inequalities in time if the data are sufficiently smooth. These estimates describe the dispersion effects and take the following form:

$$\sup_{t\in\mathbb{R}} |t|^{\frac{1}{2}} \|u(t)\|_{L^\infty(\mathbb{K})} < \infty. \tag{14}$$

This dispersion inequality is given as follows.

THEOREM 3.2. *If $u_0 \in \overset{\bullet}{\Lambda}{}^{2\alpha+4-1/2}_{1,1}(\mathbb{K})$, $u_1 \in \overset{\bullet}{\Lambda}{}^{2\alpha+4-3/2}_{1,1}(\mathbb{K})$, $f = 0$ and $u$ is solution of (1), then we have the following estimate*

$$\|u(t)\|_{L^\infty(\mathbb{K})} \leq C|t|^{-1/2}\Big\{\|u_0\|_{\overset{\bullet}{\Lambda}{}^{2\alpha+4-1/2}_{1,1}(\mathbb{K})} + \|u_1\|_{\overset{\bullet}{\Lambda}{}^{2\alpha+4-3/2}_{1,1}(\mathbb{K})}\Big\}. \tag{15}$$

*Furthermore, there exist $u_0$, $u_1 \in S_*(\mathbb{K})$ such that the solution $u$ of (1) with $f = 0$ satisfies*

$$\|u(t)\|_{L^\infty(\mathbb{K})} \geq C|t|^{-1/2}, \qquad \textit{for} \quad t \geq 1. \tag{16}$$

In order to prove the above theorem, we have to recall some classical results which remain also valid in our context. We begin by introducing some pseudo-differential operators that will be useful to our purpose.

DEFINITION 3.1. For $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$, $\delta \in \mathbb{R}$ and $t \in \mathbb{R}$, we denote by

$$b_m(\lambda) = \Big(|\lambda|\Big(m + \frac{\alpha+1}{2}\Big)\Big)^{1/2} \qquad \text{and} \qquad a_m(\lambda, t) = \frac{\sin(2b_m(\lambda)t)}{2b_m(\lambda)}$$

and we define the operators $(D_\mathbb{K})^{\delta/2}$, $A_t$ and $U_t$ as follows

$$\mathcal{F}((D_\mathbb{K})^{\delta/2} f)(\lambda, m) = (b_m(\lambda))^\delta \mathcal{F}f(\lambda, m) \tag{17}$$

$$\mathcal{F}(U_t f)(\lambda, m) = \exp(2ib_m(\lambda)t)\mathcal{F}f(\lambda, m) \tag{18}$$

$$\mathcal{F}(A_t f)(\lambda, m) = a_m(\lambda, t)t\mathcal{F}f(\lambda, m) \tag{19}$$

which allows to define the operator $\dfrac{dA_t}{dt}$ by

$$\mathcal{F}\Big(\frac{dA_t}{dt}f\Big)(\lambda, m) = \cos(2b_m(\lambda)t)\mathcal{F}f(\lambda, m). \tag{20}$$

The solution of (1) is then $u = v + w$ with

$$v(.,t) = \frac{dA_t}{dt}u_0 + A_t u_1 \tag{21}$$

and

$$w(.,t) = \int_0^t A_{t-s}f(.,s)ds. \tag{22}$$

The following results are standard and elementary.

Lemma 3.2.
**1)** *The operator* $(D_{\mathbb{K}})^{\delta/2}$ *is an isomorphism from* $\dot{\Lambda}^{\gamma}_{p,2}(\mathbb{K})$ *on* $\dot{\Lambda}^{\gamma-\delta}_{p,2}(\mathbb{K})$.
**2)** $U_t$ *is an unitary operator on* $L^2(\mathbb{K})$.
**3)** $A_t$ *is a continuous operator from* $L^2(\mathbb{K})$ *in* $\dot{\mathcal{H}}^1(\mathbb{K})$.
**4)** $\dfrac{dA_t}{dt}$ *is a continuous operator from* $L^2(\mathbb{K})$ *in* $L^2(\mathbb{K})$.
**5)** $A_0 = 0$, $\dfrac{dA_0}{dt} = Id$, $[A_t, D_{\mathbb{K}}] = 0$ *and* $\Big[\dfrac{dA_t}{dt}, D_{\mathbb{K}}\Big] = 0$.

Lemma 3.3. *For all* $p \in [1, \infty]$ *and* $t \in \mathbb{R}$ *we have the following estimates*

$$\Big\|\frac{dA_t}{dt}u_0\Big\|_{L^p} \le \frac{1}{2}\Big\{\|U_t u_0\|_{L^p} + \|U_{-t}u_0\|_{L^p}\Big\}. \tag{23}$$

$$\|A_t u_1\|_{L^p} \le \frac{1}{2}\Big\{\|(D_{\mathbb{K}})^{-1/2}(U_t u_1)\|_{L^p} + \|(D_{\mathbb{K}})^{-1/2}(U_{-t}u_1)\|_{L^p}\Big\}. \tag{24}$$

P r o o f. The results hold from the expressions of the operators $U_t$, $A_t$ and $\dfrac{dA_t}{dt}$. ■

Lemma 3.4. *Let* $\psi \in S^1_{*,0}$. *Then there exists* $C > 0$ *such that, for all* $t \in \mathbb{R}$, *we have*

$$\sup_{(y,s)\in\mathbb{K}} |U_t\psi(y,s)| \le C\min(1, |t|^{-1/2}).$$

P r o o f. We proceed as in [3] to obtain the required estimate. ∎

P r o o f o f T h e o r e m 3.2. Using Lemma 3.4, we obtain by homogeneity

$$\|U_t\psi_r\|_{L^\infty(\mathbb{K})} \leq C|t|^{-1/2}r^{-(2\alpha+4-1/2)}. \tag{25}$$

Let $\sigma \in S_*(\mathbb{K})$ such that $\mathcal{F}\sigma \in \mathcal{D}_*(\mathbb{K})$ and $\mathcal{F}\sigma = 1$ on $Supp\psi$. Then, for a suitable $f$ in $S'_*(\mathbb{K})$, we have

$$(U_tf)\#\psi_r = U_t(f\#\psi_r) = U_t(f\#\psi_r\#\sigma_r) = f\#\psi_r\#(U_t\sigma_r).$$

By Young's inequality, we obtain

$$\|(U_tf)\#\psi_r\|_{L^\infty(\mathbb{K})} = \|f\#\psi_r\|_{L^1(\mathbb{K})}\|U_t\sigma_r\|_{L^\infty(\mathbb{K})}.$$

Applying (25) to $\sigma$ it holds

$$\|(U_tf)\#\psi_r\|_{L^\infty(\mathbb{K})} \leq C|t|^{-1/2}r^{-(2\alpha+4-1/2)}\|f\#\psi_r\|_{L^1(\mathbb{K})}$$

which leads to

$$\|U_tf\|_{\dot{\Lambda}^{-1}_{\infty,1}(\mathbb{K})} \leq C|t|^{-1/2}\|f\|_{\dot{\Lambda}^{2\alpha+4-3/2}_{1,1}(\mathbb{K})}. \tag{26}$$

Combining (23) and (26) (with $f = u_0$), we obtain

$$\begin{aligned}\Big\|\frac{dA_t}{dt}u_0\Big\|_{L^\infty(\mathbb{K})} &\leq C\Big\{\|U_tu_0\|_{L^\infty(\mathbb{K})} + \|U_{-t}u_0\|_{L^\infty(\mathbb{K})}\Big\}\\ &\leq C\Big\{\|U_tu_0\|_{\dot{\Lambda}^0_{\infty,1}(\mathbb{K})} + \|U_{-t}u_0\|_{\dot{\Lambda}^0_{\infty,1}(\mathbb{K})}\Big\}.\end{aligned}$$

Now, using the fact that $(D_\mathbb{K})^{\delta/2}$ is continuous from $\dot{\Lambda}^{\gamma}_{\infty,1}(\mathbb{K})$ to $\dot{\Lambda}^{\gamma+\delta}_{\infty,1}(\mathbb{K})$ and (26), we get

$$\begin{aligned}\Big\|\frac{dA_t}{dt}u_0\Big\|_{L^\infty(\mathbb{K})} &\leq C\Big\{\|U_t(D_\mathbb{K})^{1/2}u_0\|_{\dot{\Lambda}^{-1}_{\infty,1}(\mathbb{K})} + \|U_{-t}(D_\mathbb{K})^{1/2}u_0\|_{\dot{\Lambda}^{-1}_{\infty,1}(\mathbb{K})}\Big\}\\ &\leq C|t|^{-1/2}\|(D_\mathbb{K})^{1/2}u_0\|_{\dot{\Lambda}^{2\alpha+4-3/2}_{1,1}(\mathbb{K})}\\ &= C|t|^{-1/2}\|u_0\|_{\dot{\Lambda}^{2\alpha+4-1/2}_{1,1}(\mathbb{K})}.\end{aligned}$$

On the other hand, combining (24) and (26) (with $f = u_1$), we obtain

$$\begin{aligned} \|A_t u_1\|_{L^\infty(\mathbb{K})} &\leq C\|(D_{\mathbb{K}})^{-1/2} U_t u_1\|_{L^\infty(\mathbb{K})} \\ &\leq C\|(D_{\mathbb{K}})^{-1/2} U_t u_1\|_{\dot{\Lambda}^0_{\infty,1}(\mathbb{K})} \\ &\leq C\|U_t u_1\|_{\dot{\Lambda}^{-1}_{\infty,1}(\mathbb{K})} \\ &\leq C|t|^{-1/2}\|u_1\|_{\dot{\Lambda}^{2\alpha+4-3/2}_{1,1}(\mathbb{K})}. \end{aligned}$$

And hence the solution of the homogenous equation $u = \dfrac{dA_t}{dt}u_0 + A_t u_1$ satisfies the inequality (15). To prove (16) we consider $H \in \mathcal{D}(]1/2, 2[\times\{0\})$ such that $H(1,0) = 1$. Then $u_0 = \mathcal{F}^{-1}(H)$ belongs to $S_*(\mathbb{K})$ and $u = \dfrac{dA_t}{dt}u_0$ is a solution of homogenous the following Cauchy problem

$$\begin{cases} \square_{\mathbb{K}} u &= 0 \\ u|_{t=0} &= u_0 \\ \partial_t u|_{t=0} &= 0. \end{cases}$$

In particular we have,

$$u(0, t\sqrt{\alpha+1}, t) = \int_{1/2}^{2} e^{-i\lambda t\sqrt{\alpha+1}} \cos(2t\sqrt{\lambda(\alpha+1)}) H(\lambda, 0)\lambda^{\alpha+1} d\lambda.$$

By the stationary phase's lemma (see § 7.7 of [12]) we obtain

$$u(0, t\sqrt{\alpha+1}, t) \sim \sqrt{\pi}(\alpha+1)^{-1/4} e^{-i\pi/4} t^{-1/2}; \qquad \text{if} \quad t \longrightarrow +\infty.$$

This completes the proof. ■

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